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Ising model susceptibility amplitudes I. Two-dimensional lattices

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Abstract. By combining all available information it is possible to extend our numerical knowledge of the high- and low-temperature zero-field isothermal susceptibility of the Ising model. Estimates of the third most singular term of the ferromagnetic singularity are obtained, using both direct estimates and an extension of the Generalized Law of Corresponding States. A recent conjecture of Barouch, McCoy and Wu concerning the equality of the constant term in the asymptotic expansion of the high- and low-temperature ferromagnetic susceptibility is confirmed numerically. Finally, a non-physical critical exponent of the low-temperature susceptibility of the triangular lattice is identified.

1. Introduction

While an exact analytical solution for the zero-field isothermal susceptibility of the Ising model continues to elude us, by combining the recent exact numerical results of Barouch *et al* (1973), their extension by Guttmann (1974) and the extended series obtained by Sykes *et al* (1972, 1973), it is possible to considerably advance our (numerical) knowledge of the susceptibility, both at high and low temperatures.

For the square lattice at temperatures $T > T_c$ we may write

$$\frac{kT\chi_0(T)}{m^2} \sim C_0^+(1-T_c/T)^{-7/4}+C_1^+(1-T_c/T)^{-3/4}+C_2^+(1-T_c/T)^{1/4}+\mathrm{O}[(1-T_c/T)^{5/4}]$$
$$+D_0^+ + D_1^+(1-T_c/T)+\mathrm{O}[(1-T_c/T)^2] \qquad \text{near } T\to T_c^+ \qquad (1.1a)$$

and

$$\frac{kT\chi_0(T)}{m^2} \sim E_0^+(1+T_c/T)\ln(1+T_c/T)+\text{higher-order terms} \qquad \text{near } T\to -T_c^+, \quad (1.1b)$$

where the terms with amplitudes C_i^+ represent the ferromagnetic singularity at $T = T_c^+$, the terms with amplitudes D_i^+ represent an additive function expanded around the ferromagnetic singularity, which is analytic in the physical disc $|T| \geqslant T_c$, and the term with amplitude E_0^+ represents the antiferromagnetic singularity at $T = -T_c$.

For the triangular lattice, which has no simple antiferromagnetic ordering, the situation is simpler since then $E^+ \equiv 0$, and (1.1*b*) does not apply. For the honeycomb lattice, on the other hand, there is an additional pair of singularities on the imaginary axis at $\pm iT_c$. The precise nature of this pair of singularities is not known, but following Sykes *et al* (1972) it is likely that they can be represented by allowing the following two terms to represent the square lattice susceptibility near $T = \pm iT_c$:

$$\frac{kT\chi_0(T)}{m^2} \sim F_0^+ T(1+T_c^2/T^2)^{-\eta} + G_0^+(1+T_c^2/T^2)^{-\xi}. \tag{1.2}$$

It should be borne in mind that the true nature of this singularity could be logarithmic or could contain confluent logarithmic terms.

At low temperatures, $T < T_c$, we represent the susceptibility by

$$\frac{kT\chi_0(T)}{m^2} \sim C_0^-(T_c/T-1)^{-7/4} + C_1^-(T_c/T-1)^{-3/4} + C_2^-(T_c/T-1)^{1/4} + O[(T_c/T-1)^{5/4}]$$
$$+ D_0^- + D_1^-(T_c/T-1) + O[(T_c/T-1)^2] \qquad \text{near } T \to T_c^- \tag{1.3a}$$

and

$$\frac{kT\chi_0(T)}{m^2} \sim E_0^-|T_c/T+1|^{-\theta} + E_1^-|T_c/T+1|^{1-\theta} + O[|T_c/T+1|^{2-\theta}] \qquad \text{near } T \to -T_c^-. \tag{1.3b}$$

As in the high-temperature case, the terms with amplitudes C_i^- represent the ferromagnetic singularity at $T = T_c^-$, the terms with amplitudes D_i^- represent an additive function expanded around the ferromagnetic singularity analytic in the physical disc $|T| < T_c$, but the terms with amplitudes E_i^- represent a non-physical singularity at $T = -T_c$. It should be noted that this singularity is not present for the honeycomb or square lattices but is present for the triangular lattice. The exponent θ in (1.3*b*) is not known exactly but one of the results we obtain here is $\theta \simeq 1{\cdot}25$, and we conjecture with some confidence that $\theta = \frac{5}{4}$ exactly. For the square and honeycomb lattices the above representation holds with $E_i^- \equiv 0$ so that (1.3*b*) does not apply.

In terms of the above representation, Barouch *et al* (1973) have calculated $C_0^+, C_1^+, C_0^-, C_1^-$ for the square lattice to an accuracy of 10 significant figures. By using the generalized Law of Corresponding States—introduced by Betts *et al* (1971)—Guttmann (1974) obtained C_0^+ and C_0^- for the triangular, honeycomb and Kagomé lattices to a similar accuracy, and by making a further assumption about the behaviour of the free energy, C_1^+ and C_1^- were also obtained for those lattices, to the same accuracy as the other amplitudes†. These results are summarized in table 4.

It is the purpose of the present paper to obtain numerical estimates of some of the other amplitudes and the exponent θ, defined in (1.1), (1.2) and (1.3), and in particular to test a conjecture of Barouch *et al* (1973) that $D_0^+ \equiv D_0^-$.

2. The triangular lattice

The first 16 terms of the high-temperature susceptibility series for the triangular lattice in $v = \tanh(J/kT)$ have been given by Sykes *et al* (1972), and the first 16 terms of the low-temperature series in $u = \exp(-4J/kT)$ have been given by Sykes *et al* (1973).

Taking the high-temperature series first, we form what we term a *residual series* by subtracting from the given susceptibility series the first 17 terms of the power series

† See note added in proof.

expansion of

$$C_0^+(1-T_c/T)^{-7/4}+C_1^+(1-T_c/T)^{-3/4}$$

in v, where C_0^+ and C_1^+ are as given in table 4, while

$$v_c = \tanh(J/kT_c) = 2-\sqrt{3}.$$

The residual series so obtained is denoted by $\chi_R^+(v)$ and is given below:

$$\begin{aligned}\chi_R^+(v) = \sum_{n=0}^{\infty} b_n v^n = &-0{\cdot}0228788-0{\cdot}0244312v+0{\cdot}00340892v^2+0{\cdot}0629428v^3\\ &-0{\cdot}182012v^4-1{\cdot}291493v^5-2{\cdot}476183v^6-5{\cdot}242370v^7-17{\cdot}92445v^8\\ &-62{\cdot}92120v^9-208{\cdot}3669v^{10}-686{\cdot}9611v^{11}-2300{\cdot}604v^{12}-7803{\cdot}721v^{13}\\ &-26674{\cdot}73v^{14}-91658{\cdot}23v^{15}-316409{\cdot}2v^{16}-\ldots\end{aligned} \tag{2.1}$$

From (1.1*a*) the dominant terms in $\chi_R^+(v)$ will be

$$\chi_R^+(v) \sim C_2^+(1-T_c/T)^{1/4}+D_0^+, \tag{2.2}$$

which can be written as $\chi_R^+(v) \sim A(1-v/v_c)^{1/4}+D_0^+$ where A depends on C_0^+, C_1^+ and C_2^+. This is confirmed by the behaviour of (2.1), the last four ratios being 3·3920, 3·4182, 3·4361, 3·4521, which sequence is monotonically approaching $1/v_c = 2+\sqrt{3} = 3{\cdot}7320\ldots$. To estimate D_0^+ we must evaluate $\chi_R^+(v)$ at $v = v_c$, since the term $C_2^+(1-T_c/T)^{1/4}$ of course vanishes at $T = T_c$.

The first method used to do this was to form diagonal and off-diagonal Padé approximants to $\chi_R^+(v)$ and evaluate these at v_c. The results of this calculation are summarized in table 1. It can be seen that the estimates are decreasing, so that we can only estimate $D_0^+ < -0{\cdot}038$.

Table 1. Padé approximants to $\chi_R^+(v)|_{v=v_c}$ for the triangular lattice high-temperature residual susceptibility series.

N	$[N-1/N]$	$[N/N]$	$[N+1/N]$
4	−0·02642	−0·04067	−0·03476
5	−0·03527	−0·03524	−0·03668
6	−0·03527	−0·03712	−0·03951
7	−0·03759	−0·03827	−0·03837
8	−0·03840	−0·03817	

A better method is to estimate C_2^+ and to use this to estimate the remainder in (2.1). We estimate A by writing

$$(1-v/v_c)^{1/4} = \sum_{n=0}^{\infty} a_n v^n.$$

Then the ratios b_n/a_n, where b_n is defined by (2.1), should yield a sequence that converges to A. The last six such ratios $r_n = b_n/a_n$ are 0·0339846, 0·0340422, 0·0342322, 0·0344273, 0·0345793, 0·0346955 for n = 11, 12, 13, 14, 15, 16. Extrapolating this increasing sequence against $1/n$ by forming linear extrapolants $e_n = nr_n-(n-1)r_{n-1}$ gives the following sequence: 0·034675, 0·036512, 0·036964, 0·036707, 0·036439 for n = 12, 13, 14, 15, 16. If the behaviour exhibited by the last three terms continues, we can estimate

$$A = 0{\cdot}0356 \pm 0{\cdot}002 \tag{2.3}$$

so that $C_2^+ = 0{\cdot}0690 \pm 0{\cdot}002$. Fortunately the estimate of D_0^+ given below is almost independent of this rather uncertain estimate of A.

To estimate D_0^+ we have

$$D_0^+ = \chi_R^+(v_c) = \sum_{n=0}^{\infty} b_n v_c^n \simeq \sum_{n=0}^{16} b_n v_c^n + A \sum_{n=17}^{\infty} a_n v_c^n = \sum_{n=0}^{16} b_n v_c^n - A \sum_{n=0}^{16} a_n v_c^n$$
$$= -0{\cdot}03513 - 0{\cdot}0356 \times 0{\cdot}40565 = -0{\cdot}0496 \pm 0{\cdot}002. \tag{2.4}$$

This is consistent with the upper bound $D_0^+ < -0{\cdot}038$ obtained by the Padé method. It disagrees with the result obtained by Sykes *et al* (1972) of $D_0^+ = -0{\cdot}0272$, though this disagreement is not too significant, since Sykes *et al* were more interested in showing that this constant was small rather than in determining its exact value.

Turning now to the low-temperature susceptibility series on the triangular lattice, we form the residual series by subtracting from the given series the first 16 terms of the power series expansion of

$$C_0^-(T_c/T-1)^{-7/4} + C_1^-(T_c/T-1)^{-3/4}$$

in $u = \exp(-4J/kT)$ where C_0^- and C_1^- are as given in table 4, while

$$u_c = \exp(-4J/kT_c) = \tfrac{1}{3}.$$

The residual series so obtained is denoted $\chi_R^-(u)$ and is given by

$$\chi_R^-(u) = \sum_{n=0}^{\infty} b_n u^n = -0{\cdot}00180659 - 0{\cdot}090781u - 0{\cdot}465930u^2 + 2{\cdot}09271u^3 - 7{\cdot}12483u^4$$
$$+ 22{\cdot}6766u^5 - 71{\cdot}2305u^6 + 221{\cdot}923u^7 - 687{\cdot}943u^8 + 2127{\cdot}76u^9$$
$$- 6555{\cdot}39u^{10} + 20165{\cdot}08u^{11} - 61861{\cdot}3u^{12} + 189526{\cdot}5u^{13} - 579563u^{14}$$
$$+ 1770327u^{15} - 5400352u^{16} + \ldots \tag{2.5}$$

This series alternates in sign, reflecting the dominance of the singularity on the negative real axis at $u = -u_c = -\frac{1}{3}$. To estimate D_0^- we first proceed as for the high-temperature lattice and evaluate Padé approximants to the residual series at $u = u_c$. The diagonal and off-diagonal approximants are shown in table 2. It can be seen that convergence appears to be very rapid and we estimate $D_0^- = 0{\cdot}0487 \pm 0{\cdot}001$, in excellent agreement with the

Table 2. Padé approximants to $\chi_R^-(u)|_{u=u_c}$ for the triangular lattice low-temperature residual susceptibility series.

N	$[N-1/N]$	$[N/N]$	$[N+1/N]$
4	−0·04954	−0·04895	−0·04882
5	−0·04882	−0·04877	−0·04872
6	−0·04873	−0·04870	−0·04877
7	−0·04879	−0·04866	−0·04867
8	−0·04867	−0·04867	

high-temperature result. Before discussing this result further we will estimate D_0^- by another method. From (1.3*b*) we see that

$$\chi_R^-(u) \sim E_0^-|T_c/T+1|^{-\theta} \qquad \text{near } T \to -T_c^-$$
$$\simeq B(1+u/u_c)^{-\theta} \qquad (2.6)$$

where $B = (4J/kT_c)^\theta E_0^-$. To estimate θ we may use the ratio method since estimates of $\theta - 1$ are provided by the sequence $g_n = n(-u_c r_n - 1)$, $r_n = b_n/b_{n-1}$ with b_n defined by (2.5). The last six estimates of r_n and g_n are shown in table 3, from which it can be seen

Table 3. A ratio method analysis of the non-physical singularity at $u = -u_c = -\frac{1}{3}$ for the non-physical singularity of the low-temperature triangular lattice Ising model.

n	$r_n = b_n/b_{n-1}$	$g_n = n(-u_c r_{n-1})$
11	−3·076105	0·2791
12	−3·067145	0·2710
13	−3·063733	0·2762
14	−3·057952	0·2704
15	−3·054590	0·2730
16	−3·050482	0·2692

that the estimates of g_n decrease in pairs (indicative of the weak residual singularity $C_2^-(T_c/T-1)^{1/4}$ on the positive real axis). This sequence appears to be approaching a limit around 0·25. A more careful Padé analysis (not shown) indicates quite clearly that this singularity is very close indeed to $\theta - 1 = \frac{1}{4}$ and we conjecture with some confidence that $\theta = \frac{5}{4}$ is an exact result. Assuming this to be the case, we may estimate B in (2.6) from the sequence b_n/a_n where

$$\sum_{n=0}^{\infty} a_n u^n = (1+u/u_c)^{-5/4}.$$

The last six values of b_n/a_n are −0·0558737, −0·0559694, −0·0560800, −0·0561604, −0·0562449, −0·0563115. Extrapolating these against $1/n$ yields an increasing sequence of extrapolants, so assuming that these monotonic trends continue, we estimate $B = -0{\cdot}0568 \pm 0{\cdot}0008$ or equivalently $E_0^- = -0{\cdot}0505 \pm 0{\cdot}0008$.

To estimate D_0^- we have from (1.3*a*)

$$D_0^- = \chi_R^-(u_c) = \sum_{n=0}^{\infty} b_n u_c^n \simeq \sum_{n=0}^{16} b_n u_c^n + \left(2^{-5/4} - \sum_{n=0}^{16} a_n u_c^n\right) B$$
$$= 0{\cdot}111918 + [0{\cdot}420448 - 1{\cdot}542243](-0{\cdot}0568) = -0{\cdot}0482 \pm 0{\cdot}0016.$$

This agrees well with the estimate obtained by the Padé method, $D_0^- = -0{\cdot}0487 \pm 0{\cdot}001$, and both agree with the high-temperature estimate $D_0^+ = -0{\cdot}0496 \pm 0{\cdot}002$. Now Barouch *et al* (1973) have conjectured that $D_0^+ = D_0^-$ for the square lattice, but we would expect this type of relation to be lattice-independent, so that our numerical results can be considered as evidence in support of their conjecture. Our results for the triangular lattice are summarized in table 4.

Table 4. Summary of all the amplitudes—and one exponent—obtained in this and earlier studies. Results in parentheses are obtained by an extension of the Generalized Law of Corresponding States. Those marked with an asterisk* are from Sykes *et al* (1972).

	Lattice		
Amplitude	Triangular	Square	Honeycomb
C_0^-	0·0245189020	0·0255369719	0·0277610956
C_1^-	−0·0016835479	−0·0019894107	−0·0026385047
C_2^-	$(0{\cdot}007^{+0{\cdot}002}_{-0{\cdot}001})$	$0{\cdot}0095^{+0{\cdot}0024}_{-0{\cdot}0013}$	$0{\cdot}02^{+0{\cdot}02}_{-0{\cdot}01}(0{\cdot}015^{+0{\cdot}004}_{-0{\cdot}002})$
D_0^-	$-0{\cdot}0487\pm0{\cdot}001$	$-0{\cdot}115^{+0{\cdot}005}_{-0{\cdot}01}$	$-0{\cdot}24^{+0{\cdot}02}_{-0{\cdot}03}$
E_0^-	$-0{\cdot}0505\pm0{\cdot}0008$	non-existent	non-existent
θ	$\frac{5}{4}$	non-existent	non-existent
C_0^+	0·9242069582	0·9625817322	1·0464170761
C_1^+	0·0634590701	0·0749881538	0·0994548793
C_2^+	$0{\cdot}0690\pm0{\cdot}002$	$(0{\cdot}093\pm0{\cdot}003)$	$(0{\cdot}150\pm0{\cdot}004)$
D_0^+	$-0{\cdot}0496\pm0{\cdot}002$	$-0{\cdot}11\pm0{\cdot}03$	$\simeq -0{\cdot}24$
E_0^+	non-existent	$0{\cdot}196\pm0{\cdot}002$*	$0{\cdot}182\pm0{\cdot}001$*

3. Square lattice

For the square lattice Sykes *et al* (1972) have published the first 21 terms of the high-temperature susceptibility series in $v = \tanh(J/kT)$, and Sykes *et al* (1973) have published the first 11 terms of the low-temperature susceptibility series in $u = \exp(-4J/kT)$.

Taking the high-temperature series first, we form the residual series $\chi_R^+(v)$ by subtracting the terms $C_0^+(1-T_c/T)^{-7/4}+C_1^+(1-T_c/T)^{-3/4}$, expanded in v, from the susceptibility series, using C_0^+ and C_1^+ from table 4, and $v_c = \tanh(J/kT_c) = \sqrt{2}-1$. The residual series so obtained is

$$\begin{aligned}\chi_R^+(v) = {} & -0{\cdot}119651+0{\cdot}109561v-0{\cdot}154053v^2+0{\cdot}392807v^3-0{\cdot}304042v^4+0{\cdot}739688v^5\\ & -1{\cdot}453202v^6+2{\cdot}38340v^7-4{\cdot}01553v^8+6{\cdot}31562v^9-14{\cdot}0004v^{10}\\ & +22{\cdot}5076v^{11}-63{\cdot}0653v^{12}+94{\cdot}9542v^{13}-294{\cdot}414v^{14}+397{\cdot}342v^{15}\\ & -1375{\cdot}31v^{16}+1668{\cdot}40v^{17}-6590{\cdot}31v^{18}+7249{\cdot}77v^{19}-32537{\cdot}9v^{20}\\ & +31805{\cdot}2v^{21}-\ldots \end{aligned} \qquad (3.1)$$

Table 5. Padé approximants to $\chi_R^+(v)|_{v=v_c}$ for the square lattice high-temperature residual susceptibility series.

N	$[N-1/N]$	$[N/N]$	$[N+1/N]$	Row average
4	−0·0783	−0·0771	−0·0771	−0·077
5	−0·0772	−0·0765	−0·0773	−0·077
6	−0·0775	−0·0787	−0·0762	−0·077
7	−0·0766	−0·0753	−0·0798	−0·077
8	−0·0646	−0·0869	−0·0848	−0·079
9	−0·0843	−0·0858	−0·0860	−0·085
10	−0·0860	−0·0859	−0·0859	−0·086
11	−0·0859			

The alternating signs reflect the dominance of the antiferromagnetic singularity at $v = -v_c$. Our first attempt to estimate the constant term D_0^+ was just to substitute $v = v_c$ in (3.1) using (1.1*a*). This gives a sum of terms, alternating in sign, with the negative terms being consistently larger in magnitude than the positive terms. The sum thus establishes an upper bound, which is $D_0^+ < -0{\cdot}0796$.

One method used to estimate D_0^+ was to form diagonal and off-diagonal Padé approximants to $\chi_R^+(v)$ evaluated at $v = v_c$. The results are shown in table 5. Like the triangular lattice Padé table, these estimates are also decreasing. Since little is known about the convergence properties of a Padé table, it is only possible to extrapolate such a table in a fairly crude fashion. The first method used to extrapolate the table is by comparison with the triangular lattice Padé table. There, the last estimate was about 77 % of the final value. Assuming that a similar result holds in this case gives an estimate of $D_0^+ \simeq -0{\cdot}112$. A better method is to observe that if we average each row of the Padé table, the last four row averages extrapolate roughly linearly against $1/n$, giving an estimate $D_0^+ \simeq -0{\cdot}105$. (A similar method of extrapolation was recently used—with much more regular data—by Watts (1974).) We conclude from this investigation that $D_0^+ \simeq -0{\cdot}11 \pm 0{\cdot}03$. As for the triangular lattice we note that the estimate of Sykes *et al* (1972) of $D_0^+ \simeq -0{\cdot}073$ is too high, higher even than our upper bound. For E_0^+ we have not been able to improve on the estimate of Sykes *et al* (1972) that $E_0^+ = 0{\cdot}19 \pm 0{\cdot}01$ or, assuming high–low temperature symmetry of the antiferromagnetic singularity, that $E_0^+ = 0{\cdot}196 \pm 0{\cdot}002$. For completeness this result is also quoted in table 4.

Turning now to the low-temperature series, we obtain for the residual series:

$$\chi_R^-(u) = \sum_{n=0}^{\infty} b_n u^n = -0{\cdot}00556474 - 0{\cdot}425703u - 0{\cdot}218008u^2 - 1{\cdot}47208u^3 - 2{\cdot}65479u^4 - 10{\cdot}4630u^5 - 37{\cdot}0071u^6 - 155{\cdot}867u^7 - 683{\cdot}165u^8 - 3138{\cdot}93u^9 - 14841{\cdot}6u^{10} - 71830{\cdot}3u^{11} \ldots \tag{3.2}$$

All the terms are negative and assuming this trend continues we can establish an upper bound on D_0^- by substituting $u = u_c = 3 - 2\sqrt{2}$ in (3.2). This gives $D_0^- < -0{\cdot}0995$. A better estimate may be obtained by observing that $\chi_R^-(u)$ is dominated by the term

$$C_2^-(T_c/T - 1)^{1/4} \sim A(1 - u/u_c)^{1/4} = \sum_{n=0}^{\infty} a_n u^n \qquad \text{near } T \to T_c^-.$$

Estimates of A are given by the sequence b_n/a_n, the last seven of which are:

0·055167, 0·042288, 0·037202, 0·033156, 0·030354, 0·028142, 0·026364.

Extrapolating this sequence using a Neville table gives the result $A = 0{\cdot}011^{+0{\cdot}004}_{-0{\cdot}002}$, from which follows $C_2^- = 0{\cdot}0095^{+0{\cdot}0024}_{-0{\cdot}0013}$. A better estimate of D_0^- can now be obtained by writing, from (1.3*a*),

$$\begin{aligned} D_0^- = \chi_R^-(u)|_{u=u_c} &= \sum_{n=0}^{\infty} b_n u_c^n \simeq \sum_{n=0}^{11} b_n u_c^n + A \sum_{n=12}^{\infty} a_n u_c^n \\ &= \sum_{n=0}^{11} b_n u_c^n - A \sum_{n=0}^{12} a_n u_c^n \\ &= -0{\cdot}0994 - 0{\cdot}011 \times 1{\cdot}444 = -0{\cdot}115^{+0{\cdot}005}_{-0{\cdot}01}. \end{aligned} \tag{3.3}$$

Finally, another estimate of D_0^- was obtained by evaluating Padé approximants to $\chi_R^-(u)$ at $u = u_c$. The results are shown in table 6, where it can be seen that the entries are again slowly decreasing. Extrapolating row averages against $1/n$ gives a result $D_0^- \simeq -0{\cdot}11$, in agreement with that obtained in (3.3).

Table 6. Padé approximants to $\chi_R^-(u)|_{u=u_c}$ for the square lattice low-temperature residual susceptibility series.

N	$[N-1/N]$	$[N/N]$	$[N+1/N]$
3	$-0{\cdot}0983$	$-0{\cdot}1010$	$-0{\cdot}1000$
4	$-0{\cdot}1001$	$-0{\cdot}1003$	$-0{\cdot}1008$
5	$-0{\cdot}1030$	$-0{\cdot}1014$	$-0{\cdot}1014$
6	$-0{\cdot}1014$		

Our results for the square lattice are summarized in table 4. Within the confidence limits quoted it can be seen that the conjecture of Barouch *et al*, that $D_0^+ = D_0^-$, is confirmed.

4. The honeycomb lattice

The first 32 terms of the high-temperature susceptibility series in $v = \tanh(J/kT)$ for the honeycomb lattice have been given by Sykes *et al* (1972), and the first 16 terms of the low-temperature susceptibility series in $z = \exp(-2J/kT)$ have been given by Sykes *et al* (1973). As for the othei two lattices, we first form the residual susceptibilities $\chi_R^+(v)$ and $\chi_R^-(z)$ by subtracting the two most singular terms from the given susceptibility series. The amplitudes of these singular terms are as given in table 4. The critical temperature is $v_c = \tanh(J/kT_c) = 1/\sqrt{3}$ or $z_c = \exp(-2J/kT_c) = 2-\sqrt{3}$.

The high-temperature residual series is particularly difficult to analyse since it includes contributions from four singularities, symmetrically disposed on the physical disc at $\pm v_c$ and $\pm iv_c$. We have been unable to obtain consistent estimates for any of the parameters C_2^+, E_0^+, F_0^+ or G_0^+ as defined in (1.1) and (1.2). An estimate of E_0^+ has been obtained by Sykes *et al* (1972) and this is given in table 4.

An attempt to estimate D_0^+ by forming Padé approximants to $\chi_R^+(v)$ and evaluating these at v_c (using (1.1*a*)) leads to a very erratic Padé table (not shown) which shows no steady trend and is therefore impossible to extrapolate.

Turning to the low-temperature residual series, this is

$$\chi_R^-(z) = \sum_{n=0}^{\infty} b_n z^n = -0{\cdot}00237453 - 0{\cdot}174386z - 1{\cdot}11722z^2 - 1{\cdot}69599z^3 - 2{\cdot}48557z^4 - 9{\cdot}14874z^5 - 14{\cdot}1269z^6 - 42{\cdot}2328z^7 - 104{\cdot}648z^8 - 311{\cdot}675z^9 - 895{\cdot}403z^{10} - 2725{\cdot}31z^{11} - 8401{\cdot}61z^{12} - 26476{\cdot}3z^{13} - 84536{\cdot}0z^{14} - 273398z^{15} - 893304z^{16} - \dots \quad (4.1)$$

Evaluating this at $z = z_c = 3-2\sqrt{2}$ gives an upper bound $D_0^- < -0{\cdot}209$ assuming that the terms continue to be of uniform sign. Proceeding as for the square lattice, we can obtain a better estimate by noting that $\chi_R^-(z)$ is dominated by the term

$$C_2^-(T_c/T-1)^{1/4} \sim A(1-z/z_c)^{1/4} = A\sum_{n=0}^{\infty} a_n z^n.$$

Estimates of A are given by the sequence b_n/a_n which, when extrapolated by a Neville table, yields $A = 0{\cdot}02^{+0{\cdot}02}_{-0{\cdot}01}$ or $C_2^- = 0{\cdot}02^{+0{\cdot}02}_{-0{\cdot}01}$.

An estimate of D_0^- is therefore given by

$$\chi_R^-(z_c) = D_0^- = \sum_{n=0}^{\infty} b_n z_c^n \simeq \sum_{n=0}^{16} b_n z_c^n + A\sum_{n=17}^{\infty} a_n z_c^n$$

$$= \sum_{n=0}^{16} b_n z_c^n - A\sum_{n=0}^{16} a_n z_c^n$$

$$= -0{\cdot}2092 - 0{\cdot}02\times 1{\cdot}4056 = -0{\cdot}24^{+0{\cdot}02}_{-0{\cdot}03}. \qquad (4.2)$$

An alternative method of estimating D_0^- is just to form Padé approximants to $\chi_R^-(z)$ and evaluate these at $z = z_c$. The Padé approximants so obtained are shown in table 7.

Table 7. Padé approximants to $\chi_R^-(z)|_{z=z_c}$ for the honeycomb lattice low-temperature residual susceptibility series.

N	$[N-1/N]$	$[N/N]$	$[N+1/N]$
4	−0·2013	−0·2061	−0·1912
5	−0·0846	−0·2185	−0·2114
6	−0·2095	−0·2121	−0·2131
7	−0·2126	−0·2154	−0·2149
8	−0·2149	−0·2151	

They are seen to be slowly decreasing, and extrapolating row averages against $1/N$ yields $D_0^- \simeq -0{\cdot}23$, in agreement with the result obtained in (4.2). Since $D_0^+ = D_0^-$ was found to hold for both the square and triangular lattices (within the confidence limits quoted) it is almost certainly true for the honeycomb lattice as well. In that case we can write $D_0^+ \simeq -0{\cdot}24$.

Our results for the honeycomb lattice are also summarized in table 4. In § 5 we consider these results in the light of the Generalized Law of Corresponding States.

5. The generalized law of corresponding states

This law, first enunciated by Betts *et al* (1971), and recently extended by Ritchie and Betts (1975), asserts that for a number of lattice models including the Ising model *the most singular part* of the free energy per site on lattice X, $f_X(t_X, h_X)$ is related to the most singular part of the free energy per site on lattice Y, $f_Y(t_Y, h_Y)$ by the relation

$$n_X f_X(t_X, h_X) = n_Y f_Y(t_Y, h_Y) = f(t, h). \qquad (5.1)$$

Here the reduced magnetic field variable $h = \mu H/kT$ (in the usual notation) is scaled according to

$$n_X h_X = n_Y h_Y = h \qquad (5.2)$$

and the reduced temperature $t = T/T_c - 1$ is scaled according to

$$g_X t_X = g_Y t_Y = t. \tag{5.3}$$

This generalized law is also known as 'lattice–lattice scaling'. The parameters n_X and g_X are lattice-dependent and are shown in table 8. From (5.1) it follows that the field

Table 8. Critical scaling parameters g_X and n_X for the common planar Ising lattices.

Lattice	g_X	n_X
Triangular	1·0000000000	1·0000000000
Square	1·1345681212	1·2990381057
Honeycomb	1·3841935033	2·0000000000

derivatives of the free energy are related by

$$n_Y^{l-1}\frac{\partial^l f_X}{\partial h_X^l} = n_X^{l-1}\frac{\partial^l f_Y}{\partial h_Y^l}, \tag{5.4}$$

while in zero field the derivatives behave like

$$\begin{aligned}\frac{\partial^l f_X}{\partial h_X^l} &= G_{l,X}^{+} t_X^{-\gamma_l} \qquad l \text{ even}, t > 0 \\ &= G_{l,X}^{-}(t_X)^{-\gamma_l'} \qquad t < 0.\end{aligned} \tag{5.5}$$

Combining the above equations we obtain

$$\frac{G_{2l,X}^{+}}{G_{2l,Y}^{+}} = \left(\frac{n_X}{n_Y}\right)^{2l-1}\left(\frac{g_X}{g_Y}\right)^{-\gamma_{2l}} \qquad t > 0$$

and

$$\frac{G_{l,X}^{-}}{G_{l,Y}^{-}} = \left(\frac{n_X}{n_Y}\right)^{l-1}\left(\frac{g_X}{g_Y}\right)^{-\gamma_l'} \qquad t < 0. \tag{5.6}$$

Using these results the amplitudes C_0^+ and C_0^- for the triangular, Kagomé and honeycomb lattices were obtained by Guttmann (1974) from the square lattice values of Barouch *et al* (1973).

To obtain the amplitudes C_1^+ and C_1^- it is necessary to assume that the next most singular term in the free energy also scales, that is, (5.1) holds with f replaced by the most singular term plus the next most singular term. In this way the results for C_1^+ and C_1^- for the triangular and honeycomb lattices were also obtained (Guttmann 1974). It would be nice to estimate C_2^+ and C_2^- for all lattices from the given estimates but this involves the assumption that (5.1) holds for the most singular term plus the next two most singular terms. As we shall show by considering the exact results for the spontaneous magnetization, this does not appear to be the case.

The square lattice Ising model spontaneous magnetization is given by Yang (1952) as

$$I^8(u) = (1+u)^2(1-u)^{-4}(u_c-u)(1/u_c-u) \tag{5.6}$$

where $u = \exp(-4J/kT_c)$ and $u_c = 3-2\sqrt{2}$. Writing this as

$$I(T) = B_0(T_c/T-1)^{1/8} + B_1(T_c/T-1)^{9/8} + B_2(T_c/T-1)^{17/8} + \ldots, \tag{5.7}$$

the amplitudes B_0, B_1, B_2 may be obtained by tedious but straightforward algebraic manipulation. The results of this calculation, with the corresponding results for the triangular and honeycomb lattices, are shown in table 9. According to the generalized

Table 9. Exact spontaneous magnetization amplitudes for the common planar Ising lattices. Formation of the quantity B_0B_2/B_1^2 illustrates the (slight) breakdown of lattice–lattice scaling for the third most singular term of the free energy.

	Triangular	Square	Honeycomb
B_0	$2^{1/2}K_C^{1/8}$	$2^{7/16}K_C^{1/8}$	$\surd(2)3^{-1/16}K_C^{1/8}$
B_1	$-9\surd(2)K_C^{9/8}/8$	$-9\surd(2) . 2^{7/16}K_C^{9/8}/16$	$-3\surd(6)K_C^{9/8}/2^3 . 3^{1/16}$
B_2	$1259\surd(2)K_C^{17/8}/384$	$1243 . 2^{7/16}K_C^{17/8}/3 . 2^8$	$1227\surd(2)K_C^{17/8}/9 . 2^7 . 3^{1/16}$
K_C	0·274653072167	0·440686793510	0·658478948462
B_0/B_1	$-8/9K_C$	$-8\surd(2)/9K_C$	$-8\surd(3)/9K_C$
B_0B_2/B_1^2	$\frac{1259}{486}$	$\frac{1243}{486}$	$\frac{1227}{486}$

law of corresponding states, assuming its extension to the next most singular term of the free energy, we can write

$$\frac{B_{0,X}}{B_{0,Y}}\frac{B_{1,Y}}{B_{1,X}} = \frac{g_Y}{g_X}. \tag{5.8}$$

For the triangular (X) and square (Y) lattices the left-hand side is given by 1·1345681212 while the right-hand side is, from table 8, given by 1·1345681212. Similarly, for the square (X) and honeycomb (Y) lattice we have left-hand side = 1·220017976 = right-hand side, thus confirming lattice–lattice scaling for the next most singular term also. If we try to extend this to the third most singular term we obtain

$$\frac{B_{1,X}}{B_{1,Y}}\frac{B_{2,Y}}{B_{2,X}} = \frac{g_Y}{g_X}. \tag{5.9}$$

Combining this with (5.8) we obtain

$$\frac{B_{0,X}B_{2,X}}{B_{1,X}^2} = \frac{B_{0,Y}B_{2,Y}}{B_{1,Y}^2}. \tag{5.10}$$

That is, B_0B_2/B_1^2 should be lattice-independent, or an *invariant*. The quantity B_0B_2/B_1^2 is also tabulated in table 9 and it can be seen that it is *not* constant, as predicted by (5.10). The square lattice value of B_0B_2/B_1^2 differs from both the triangular and honeycomb lattices by 1·3 %, so that the lattice–lattice scaling hypothesis breaks down for the third most singular term of the free energy by a small amount.

Now the susceptibility amplitudes C_2^+ and C_2^- that we have estimated, notably C_2^+ for the triangular lattice and C_2^- for the square and honeycomb lattices, have associated uncertainties considerably greater than 1·3 %. We may therefore assume that lattice–lattice scaling for the third most singular term holds within the accuracy of this calculation, and we estimate C_2^+ and C_2^- for the other lattices using the relations

$$\frac{C_{0,X}^+C_{2,X}^+}{(C_{1,X}^+)^2} = \text{constant}, \qquad \frac{C_{0,X}^-C_{2,X}^-}{(C_{1,X}^-)^2} = \text{constant}. \tag{5.11}$$

Taking $C^{+}_{2,\mathrm{tri}} = 0{\cdot}0690 \pm 0{\cdot}002$ from table 4, we estimate from (5.11) $C^{+}_{2,\mathrm{sq}} = 0{\cdot}093 \pm 0{\cdot}003$ and $C^{+}_{2,\mathrm{hc}} = 0{\cdot}150 \pm 0{\cdot}004$. For $T < T_c$ we take $C^{-}_{2,\mathrm{sq}} = 0{\cdot}0095^{+0{\cdot}0024}_{-0{\cdot}0013}$ as the most accurate estimate of C^{-}_{2}, and from (5.11) we estimate $C^{-}_{2,\mathrm{tri}} = 0{\cdot}007^{+0{\cdot}002}_{-0{\cdot}001}$ and $C^{-}_{2,\mathrm{hc}} = 0{\cdot}015^{+0{\cdot}004}_{-0{\cdot}002}$, compared to the direct estimate of $C^{-}_{2,\mathrm{hc}} = 0{\cdot}02^{+0{\cdot}02}_{-0{\cdot}01}$. These new results are shown in table 4, in parentheses, to indicate that they are not as reliable as the other results shown in the table. The above results for C^{+}_{2} and C^{-}_{2} could also have been derived from the relation

$$\frac{C^{+}_{2,\mathrm{X}}}{C^{+}_{2,\mathrm{Y}}} = \frac{C^{-}_{2,\mathrm{X}}}{C^{-}_{2,\mathrm{Y}}} = \left(\frac{n_{\mathrm{X}}}{n_{\mathrm{Y}}}\right)\left(\frac{g_{\mathrm{X}}}{g_{\mathrm{Y}}}\right)^{2-\gamma}. \tag{5.12}$$

We emphasize that this expression and (5.11), by analogy with the exact results of the spontaneous magnetization, are likely to be in error by about 1% and we do not suggest they are exact. Rather, they appear to be sufficiently close to exactness to be useful.

6. Discussion

It has been our intention to derive as much numerical information as possible about the zero-field isothermal susceptibility of the two-dimensional Ising model on the three most common planar lattices. The two leading amplitude terms at both high and low temperatures have been derived in an earlier paper (Guttmann 1974). In this paper we have derived the next amplitude term, and the constant term for all three lattices. We have confirmed—within the numerical uncertainty of our estimates—the conjecture made by Barouch *et al* (1972) that the constant term is equal at high and low temperatures. It is noteworthy in this context that Sykes *et al* (1972) obtained a similar result for the *anti*ferromagnetic susceptibility; that is, they found numerical agreement for the constant term in the asymptotic expansion of the antiferromagnetic susceptibility at high and low temperatures for both the square and honeycomb lattices. The triangular lattice of course has no simple antiferromagnetic ordered state.

Finally we have estimated the critical exponent corresponding to the non-physical singularity of the triangular lattice low-temperature susceptibility at $u = -u_c = -\frac{1}{3}$, and have found it to be approximately equal to $\frac{5}{4}$, corresponding to a divergence.

In the following paper we attempt to carry over as much of this work as possible to the three-dimensional Ising model.

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Note added in proof. Since this paper was accepted, the paper of Ritchie and Betts (1975) has appeared, and overlaps some of this work. Specifically, they show that lattice–lattice scaling extends to the second most singular term only if the temperature and field scaling variables are as given below (5.1) and (5.2), and thus differ from those used in the original lattice–lattice scaling theory. Further, they show that the next most singular term of the free energy does not scale for the Kagomé lattice. This may be connected with the fact that, for the Kagomé lattice, each lattice site is surrounded by tesalations of 3 and 6 sides, while for the square, triangular and honeycomb lattices, the tessalations surrounding each lattice site are all of the same size.

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